Moment of Inertia

• In deriving the equations of motion, we defined the moment of inertia as

\[ I_o = \sum_{i} m_i r_i^2 \]

where \( m_i \) is the mass of the \( i \)th particle and \( r_i \) is the perpendicular distance from the axis \( L_o \) to the \( i \)th particle

• For a continuous distribution of mass, the moment of inertia about an axis \( L_o \) would be

\[ I_o = \int r^2 dm \]
In deriving the moment of inertia for a 1D system, we define an elemental mass $dm$ using an elemental length $dr$.

For example, for a slender bar:

We define an elemental mass

\[ dm = \rho A \, dr \]

Where $\rho$ is the density of material, therefore the Moment of Inertia about a perpendicular axis through its center of mass would be

\[ I_o = \int r^2 \, dm \]

\[ \int_{-\frac{l}{2}}^{\frac{l}{2}} \rho Ar^2 \, dr = \frac{1}{12} \rho Al^3 \]

Since the mass of the bar $m = \rho Al$, we can write

\[ I_o = \frac{1}{12} ml^2 \]
In deriving the moment of inertia for a 2D system, we define an elemental mass \( dm \) using an elemental area \( dA \).

For example, for a thin plate:

We define an elemental mass

\[
dm = \rho T \, dA
\]

Where \( \rho \) is the density of material, and \( T \) is the thickness, therefore the Moment of Inertia about a perpendicular axis through its center of mass would be

\[
I_o = \int r^2 \, dm
= \int \rho T r^2 \, dA = \rho T \int r^2 \, dA
\]

The integral on the right is called the polar moment of inertia \( J_o \) of the cross sectional area.

We can write the moment of inertia of the plate about the \( z \)-axis

\[
I_o = \frac{m}{A} J_o
\]
2-D systems-perpendicular axis

The moment of inertia about the x-axis would be given by,

\[ I_x = \int y^2 \, dm = \rho T \int y^2 \, dA \]

While the moment of inertia about the y-axis would be

\[ I_y = \int x^2 \, dm = \rho T \int x^2 \, dA \]

Since \( r^2 = x^2 + y^2 \)

\[ dm = \rho T \, dA \]

We see that

\[ I_z = \rho T \int r^2 \, dA \]

\[ = \rho T \int x^2 + y^2 \, dA \]

\[ = I_x + I_y \]

Figure: 18-10 - (a) A plate of arbitrary shape and uniform thickness \( T \). (b) An element of volume obtained by projecting an element of area \( dA \) through the plate.

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Suppose we know the moment of inertia $I$ about a line $L$ that goes through the center of mass of an object and we wish to determine its moment of inertia $I_O$ about a line $L_O$ parallel to line $L$.
The moment of inertia about line $L_0$ is given by,

$$I_0 = \int r^2 dm = \int x^2 + y^2 \, dm$$

Let line $L$ pass through the origin of co-ordinate system $x'y'$ through the center of mass such that

$$x = x' + dx \quad \text{and} \quad y = y' + dy$$

The moment of inertia about a line $L_0$ could then be written as

$$I_0 = \int x'^2 + y'^2 \, dm + 2dx \int x' \, dm + 2dy \int y' \, dm + \int d_x^2 + d_y^2 \, dm$$
Recall that the coordinates of the center of mass are defined as

\[
\bar{x} = \frac{\int x' dm}{\int dm}
\]

\[
\bar{y} = \frac{\int y' dm}{\int dm}
\]

\[
I_o = \int x'^2 + y'^2 \, dm + 2d_x \int x' dm +
2d_y \int y' dm + \int d_x^2 + d_y^2 \, dm = 0
\]

\[
I_o = I + md^2
\]
Example 18.6A

**Example 18.6  Moment of Inertia of an L-Shaped Bar** *(Related Problem 18.72)*

Two homogeneous slender bars, each of length \( l \), mass \( m \), and cross-sectional area \( A \), are welded together to form an L-shaped object. Use integration to determine the moment of inertia of the object about the axis \( L_O \) through point \( O \). The axis \( L_O \) is perpendicular to the two bars.

**Strategy**

Using the same integration procedure we used for a single bar, we can determine the moment of inertia of each bar about \( L_O \) and sum the results.
Example 18.6  Moment of Inertia of an L-Shaped Bar (Continued)

Solution
We orient a coordinate system with the z axis along $L_O$ and the x axis collinear with bar 1 (Fig. a). The mass of the differential element of length $dx$ of bar 1 is $dm = \rho A \, dx$. The moment of inertia of bar 1 about $L_O$ is

$$(I_O)_1 = \int r^2 \, dm = \int_0^l \rho A x^2 \, dx = \frac{1}{3} \rho Al^3.$$ 

In terms of the mass of the bar, $m = \rho Al$, we can write this result as

$$(I_O)_1 = \frac{1}{3}ml^2.$$ 

The mass of the element of length $dy$ of bar 2 shown in Fig. b is $dm = \rho A \, dy$. From the figure, we see that the perpendicular distance from $L_O$ to the element is $r = \sqrt{l^2 + y^2}$. Therefore, the moment of inertia of bar 2 about $L_O$ is

$$(I_O)_2 = \int r^2 \, dm = \int_0^l \rho A(l^2 + y^2) \, dy = \frac{4}{3} \rho Al^3.$$ 

In terms of the mass of the bar, we obtain

$$(I_O)_2 = \frac{4}{3}ml^2.$$ 

The moment of inertia of the L-shaped object about $L_O$ is

$$I_O = (I_O)_1 + (I_O)_2 = \frac{1}{3}ml^2 + \frac{4}{3}ml^2 = \frac{5}{3}ml^2.$$ 

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Example 18.6C

Example 18.6  Moment of Inertia of an L-Shaped Bar (Continued)

Critical Thinking
In this example we used integration to determine a moment of inertia of an object consisting of two straight bars. The same procedure could be applied to more complicated objects made of such bars, but it would obviously be cumbersome. Once we have used integration to determine a moment of inertia of a single bar, such as Eq. (18.22), it would be very convenient to be able to use that result to determine moments of inertia of composite objects made of bars without having to resort to integration. We show how this can be done in the next section.
Example 18.7  Moments of Inertia of a Triangular Plate (Related Problem 18.76)

The thin, homogeneous plate is of uniform thickness and mass \( m \). Determine its moments of inertia about the \( x \), \( y \), and \( z \) axes.

**Strategy**
The moments of inertia about the \( x \) and \( y \) axes are given by Eqs. (18.24) and (18.25) in terms of the moments of inertia of the cross-sectional area of the plate. We can determine the moment of inertia of the plate about the \( z \) axis from Eq. (18.26).

**Solution**
From Appendix B, the moments of inertia of the triangular area about the \( x \) and \( y \) axes are \( I_x = \frac{1}{12}bh^3 \) and \( I_y = \frac{1}{4}hb^3 \). Therefore, the moments of inertia of the plate about the \( x \) and \( y \) axes are

\[
I_{x\text{ axis}} = \frac{m}{A} I_x = \frac{m}{\frac{1}{2}bh} \left( \frac{1}{12}bh^3 \right) = \frac{1}{6}mh^2
\]

and

\[
I_{y\text{ axis}} = \frac{m}{A} I_y = \frac{m}{\frac{1}{2}bh} \left( \frac{1}{4}hb^3 \right) = \frac{1}{2}mb^2.
\]

The moment of inertia about the \( z \) axis is

\[
I_{z\text{ axis}} = I_{x\text{ axis}} + I_{y\text{ axis}} = m\left( \frac{1}{6}h^2 + \frac{1}{2}b^2 \right).
\]

**Critical Thinking**
As this example demonstrates, you can use the moments of inertia of areas tabulated in Appendix B to determine moments of inertia of thin homogeneous plates. For plates with more complicated shapes, you can use the methods for determining moments of inertia of composite areas.
\[ J_x = \int y^2 dA \]
\[ = \int_0^h y^2 (b - x) dy \]
\[ = \int_0^h y^2 (b - \frac{by}{h}) dy \]
\[ = \int_0^h y^2 \frac{b}{h} (h - y) dy \]
\[ J_y = \frac{1}{12} bh^3 \]

\[ J_y = \int x^2 dA \]
\[ = \int_0^b x^2 y dx \]
\[ = \int_0^b x^2 \frac{hx}{b} dx \]
\[ J_x = \frac{bh^3}{4} \]
Example 18.8A

**Application of the Parallel-Axis Theorem** *(Related Problems 18.82, 18.83)*

Two homogeneous, slender bars, each of length \( l \) and mass \( m \), are welded together to form an L-shaped object. Determine the moment of inertia of the object about the axis \( L_O \) through point \( O \). (The axis \( L_O \) is perpendicular to the two bars.)

**Strategy**

The moment of inertia of a straight slender bar about a perpendicular axis through its center of mass is given by Eq. (18.22). We can use the parallel-axis theorem to determine the moments of inertia of the bars about the axis \( L_O \) and sum them to obtain the moment of inertia of the composite bar.
Example 18.8 Application of the Parallel-Axis Theorem (Continued)

Solution

Choose the Parts  The parts are the two bars, which we call bar 1 and bar 2 (Fig. a).

Determine the Moments of Inertia of the Parts  From Appendix C, the moment of inertia of each bar about a perpendicular axis through its center of mass is $I = \frac{1}{12} ml^2$. The distance from $L_O$ to the parallel axis through the center of mass of bar 1 is $\frac{1}{2}l$ (Fig. a). Therefore, the moment of inertia of bar 1 about $L_O$ is

$$(I_O)_1 = I + d^2m = \frac{1}{12} ml^2 + \left(\frac{1}{2}l\right)^2 m = \frac{3}{4} ml^2.$$

The distance from $L_O$ to the parallel axis through the center of mass of bar 2 is $[l^2 + \left(\frac{1}{2}l\right)^2]^{1/2}$. The moment of inertia of bar 2 about $L_O$ is

$$(I_O)_2 = I + d^2m = \frac{1}{12} ml^2 + \left[l^2 + \left(\frac{1}{2}l\right)^2\right] m = \frac{4}{3} ml^2.$$
Sum the Results  The moment of inertia of the L-shaped object about $L_O$ is

$$I_O = (I_o)_1 + (I_o)_2 = \frac{1}{3}ml^2 + \frac{4}{3}ml^2 = \frac{5}{3}ml^2.$$  

Critical Thinking
Compare this solution with Example 18.6, in which we used integration to determine the moment of inertia of the same object about $L_O$. We obtained the result much more easily with the parallel-axis theorem, but of course we needed to know the moments of inertia of the bars about the axes through their centers of mass.
Example 18.9  
Moments of Inertia of a Composite Object  
(Related Problems 18.102, 18.103)

The object consists of a slender 3-kg bar welded to a thin, circular 2-kg disk. Determine the moment of inertia of the object about the axis $L$ through its center of mass. (The axis $L$ is perpendicular to the bar and disk.)

Strategy
We must locate the center of mass of the composite object and then apply the parallel-axis theorem. We can obtain the moments of inertia of the bar and disk from Appendix C.

Solution
Choose the Parts  The parts are the bar and the disk. Introducing the coordinate system in Fig. a, we have, for the $x$ coordinate of the center of mass of the composite object,

$$
\bar{x} = \frac{\bar{x}_{\text{bar}} m_{\text{bar}} + \bar{x}_{\text{disk}} m_{\text{disk}}}{m_{\text{bar}} + m_{\text{disk}}}
$$

$$=
\frac{(0.3 \text{ m})(3 \text{ kg}) + (0.6 \text{ m} + 0.2 \text{ m})(2 \text{ kg})}{(3 \text{ kg}) + (2 \text{ kg})} = 0.5 \text{ m}.
$$

(a) The coordinate $\bar{x}$ of the center of mass of the object.
Example 18.9 Moments of Inertia of a Composite Object (Continued)

Determine the Moments of Inertia of the Parts The distance from the center of mass of the bar to the center of mass of the composite object is 0.2 m (Fig. b). Therefore, the moment of inertia of the bar about \( L \) is

\[
I_{\text{bar}} = \frac{1}{12} (3 \text{ kg})(0.6 \text{ m})^2 + (3 \text{ kg})(0.2 \text{ m})^2 = 0.210 \text{ kg-m}^2.
\]

The distance from the center of mass of the disk to the center of mass of the composite object is 0.3 m (Fig. c). The moment of inertia of the disk about \( L \) is

\[
I_{\text{disk}} = \frac{1}{2} (2 \text{ kg})(0.2 \text{ m})^2 + (2 \text{ kg})(0.3 \text{ m})^2 = 0.220 \text{ kg-m}^2.
\]

Sum the Results The moment of inertia of the composite object about \( L \) is

\[
I = I_{\text{bar}} + I_{\text{disk}} = 0.430 \text{ kg-m}^2.
\]

Critical Thinking
This example demonstrates the most common procedure for determining moments of inertia of objects in engineering applications. Objects usually consist of assemblies of parts. The center of mass of each part and its moment of inertia about the axis through its center of mass must be determined. (It may be necessary to determine this information experimentally, or it is sometimes supplied by manufacturers of subassemblies.) Then the center of mass of the composite object is determined, and the parallel axis theorem is used to determine the moment of inertia of each part about the axis through the center of mass of the composite object. Finally, the individual moments of inertia are summed to obtain the moment of inertia of the composite object.
Figure: 19-05 - Kinetic energy in general planar motion.

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\[ T = \frac{1}{2} m v^2 + \frac{1}{2} I \omega^2 \]
Results 19.1A

**RESULTS Section 19.1**

**Principle of Work and Energy**

Let $T$ be the total kinetic energy of a system of rigid bodies. The *principle of work and energy* states that the work done by external and internal forces and couples as the system moves between two positions equals the change in the total kinetic energy of the system.

$$U_{12} = T_2 - T_1. \quad (19.5)$$
Figure: 19-06 - (a) Velocity of the center of mass. (b) Kinetic energy of a rigid body rotating about a fixed axis.

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RESULTS Section 19.1 (Continued)

Kinetic Energy

The kinetic energy of a rigid body in general planar motion is the sum of the translational kinetic energy and the rotational kinetic energy.

\[ T = \frac{1}{2} mv^2 + \frac{1}{2} I\omega^2. \]  \hspace{1cm} (19.11)

Kinetic energy of a rigid body undergoing planar motion about a fixed axis \( O \). \( I_O \) is the moment of interia of the rigid body about \( O \).

\[ T = \frac{1}{2} I_O \omega^2. \]  \hspace{1cm} (19.12)
Work done on a rigid body by a force \( \mathbf{F} \) as the point of application of the force moves from position \( (r_p)_1 \) to position \( (r_p)_2 \). A force \( \mathbf{F} \) is conservative if a potential energy \( V \) exists such that
\[
\mathbf{F} \cdot d\mathbf{r}_p = -dV. \tag{19.14}
\]

\[
U_{12} = \int_{(r_p)_1}^{(r_p)_2} \mathbf{F} \cdot d\mathbf{r}_p. \tag{19.13}
\]

\[
U_{12} = \int_{V_1}^{V_2} -dV = -(V_2 - V_1).
\]
Figure: 19-08 - (a) A rigid body subjected to a couple. (b) An equivalent couple consisting of two forces: $DF = M$ (c) Determining the work done by the forces.

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Work Done by a Couple

Work done on a rigid body by a couple $M$ as the rigid body rotates from angular position (in radians) $\theta_1$ to angular position $\theta_2$. A couple $M$ is conservative if a potential energy $V$ exists such that

$$M \, d\theta = -dV.$$  \hspace{1cm} (19.16)

$$U_{12} = \int_{\theta_1}^{\theta_2} M \, d\theta.$$  \hspace{1cm} (19.15)

If $M$ is constant,

$$U_{12} = M (\theta_2 - \theta_1).$$

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Figure: 19-09 - (a) A linear torsional spring connected to a bar. (b) The spring exerts a couple of magnitude \( k\theta \) in the direction opposite that of the bar’s rotation.
Example 19.2  Applying Work and Energy to a Motorcycle (Related Problems 19.27, 19.28)

Each wheel of the motorcycle has mass \( m_w = 9 \text{ kg} \), radius \( R = 330 \text{ mm} \), and moment of inertia \( I = 0.8 \text{ kg-m}^2 \). The combined mass of the rider and the motorcycle, not including the wheels, is \( m_C = 142 \text{ kg} \). The motorcycle starts from rest, and its engine exerts a constant couple \( M = 140 \text{ N-m} \) on the rear wheel. Assume that the wheels do not slip. What horizontal distance \( b \) must the motorcycle travel to reach a velocity of 25 m/s?
Example 19.2 Applying Work and Energy to a Motorcycle (Continued)

**Strategy**
We can apply the principle of work and energy to the system consisting of the rider and the motorcycle, including its wheels, to determine the distance \( b \).

**Solution**
Determining the distance \( b \) requires three steps.

**Identify the Forces and Couples that Do Work** We draw the free-body diagram of the system in Fig. a. The weights do no work because the motion is horizontal, and the forces exerted on the wheels by the road do no work because the velocity of their point of application is zero. (See Active Example 19.1.) Thus, no work is done by external forces and couples! However, work is done by the couple \( M \) exerted on the rear wheel by the engine (Fig. b). Although this is an internal couple for the system we are considering—the wheel exerts an opposite couple on the body of the motorcycle—net work is done because the wheel rotates whereas the body does not.

(a) Free-body diagram of the system.
Example 19.2 Applying Work and Energy to a Motorcycle (Continued)

(b) Isolating the rear wheel.

Apply Work and Energy If the motorcycle moves a horizontal distance $b$, the wheels turn through an angle $b/R$ rad, and the work done by the constant couple $M$ is

$$U_{12} = M(\theta_2 - \theta_1) = M\left(\frac{b}{R}\right).$$

Let $v$ be the motorcycle’s velocity and $\omega$ the angular velocity of the wheels when the motorcycle has moved a distance $b$. The work equals the change in the total kinetic energy:

$$M\left(\frac{b}{R}\right) = \frac{1}{2} m_c v^2 + 2\left(\frac{1}{2} m_w v^2 + \frac{1}{2} I\omega^2\right) - 0. \quad (1)$$
Example 19.2  Applying Work and Energy to a Motorcycle (Continued)

Determine Kinematic Relationship  The angular velocity of the rolling wheels is related to the velocity \( v \) by \( \omega = v/R \). Substituting this relation into Eq. (1) and solving for \( b \), we obtain

\[
b = \left( \frac{1}{2} m_C + m_W + \frac{I}{R^2} \right) \frac{Rv^2}{M}
\]

\[
= \left[ \frac{1}{2} (142 \text{ kg}) + (9 \text{ kg}) + \frac{0.8 \text{ kg} \cdot \text{m}^2}{(0.33 \text{ m})^2} \right] \frac{0.33 \text{ m} \cdot (25 \text{ m/s})^2}{140 \text{ N} \cdot \text{m}}
\]

\[
= 129 \text{ m}.
\]

Critical Thinking  Although we drew separate free-body diagrams of the motorcycle and its rear wheel to clarify the work done by the couple exerted by the engine, notice that we treated the motorcycle, including its wheels, as a single system in applying the principle of work and energy. By doing so, we did not need to consider the work done by the internal forces between the motorcycle’s body and its wheels. When applying the principle of work and energy to a system of rigid bodies, you will usually find it simplest to express the principle for the system as a whole. This is in contrast to determining the motion of a system of rigid bodies by using the equations of motion, which usually requires that you draw free-body diagrams of each rigid body and apply the equations to them individually.
Example 19.3  Applying Conservation of Energy to a Linkage  (Related Problem 19.40)

The slender bars $AB$ and $BC$ of the linkage have mass $m$ and length $l$, and the collar $C$ has mass $m_C$. A torsional spring at $A$ exerts a clockwise couple $k\theta$ on bar $AB$. The system is released from rest in the position $\theta = 0$ and allowed to fall. Neglecting friction, determine the angular velocity $\omega = d\theta/dt$ of bar $AB$ as a function of $\theta$.

Strategy
The objective in this example—determining an angular velocity as a function of the position of the system—encourages an energy approach. We must first identify forces and couples that do work on the system. If they are conservative, we can apply conservation of energy to determine $\omega$ as a function of $\theta$. If non-conservative forces do work on the system, we can apply the principle of work and energy.

Solution
Identify the Forces and Couples that Do Work  We draw the free-body diagram of the system in Fig. a. The forces and couples that do work—the weights of the bars and collar and the couple exerted by the torsional spring—are conservative. We can use conservation of energy and the kinematic relationships between the angular velocities of the bars and the velocity of the collar to determine $\omega$ as a function of $\theta$. 
**Example 19.3 Applying Conservation of Energy to a Linkage (Continued)**

**Apply Conservation of Energy** We denote the center of mass of bar $BC$ by $G$ and the angular velocity of bar $BC$ by $\omega_{BC}$ (Fig. b). The moment of inertia of each bar about its center of mass is $I = \frac{1}{12} ml^2$. Since bar $AB$ rotates about the fixed point $A$, we can write its kinetic energy as

$$T_{\text{bar } AB} = \frac{1}{2} I\omega^2 = \frac{1}{2} \left[I + \left(\frac{l}{2}\right)^2 m\right] \omega^2 = \frac{1}{6} ml^2 \omega^2.$$  

The kinetic energy of bar $BC$ is

$$T_{\text{bar } BC} = \frac{1}{2} mv_G^2 + \frac{1}{2} I\omega_{BC}^2 = \frac{1}{2} mv_G^2 + \frac{1}{24} ml^2 \omega_{BC}^2.$$  

The kinetic energy of the collar $C$ is

$$T_{\text{collar}} = \frac{1}{2} m_C v_C^2.$$  

Using the datum in Fig. (a), we obtain the potential energies of the weights:

$$V_{\text{bar } AB} + V_{\text{bar } BC} + V_{\text{collar}} = mg\left(\frac{1}{2} l \cos \theta\right) + mg\left(\frac{l}{2} \cos \theta\right) + m_C g (2l \cos \theta).$$  

The potential energy of the torsional spring is given by Eq. (19.17):

$$V_{\text{spring}} = \frac{1}{2} k \theta^2.$$  

We now have all the ingredients to apply conservation of energy. We equate the sum of the kinetic and potential energies at the position $\theta = 0$ to the sum of the kinetic and potential energies at an arbitrary value of $\theta$:

$$T_1 + V_1 = T_2 + V_2;$$  

$$0 + 2mg l + 2m_C g l = \frac{1}{6} ml^2 \omega^2 + \frac{1}{2} mv_G^2 + \frac{1}{24} ml^2 \omega_{BC}^2 + \frac{1}{2} m_C v_C^2 + 2mg l \cos \theta + 2m_C g l \cos \theta + \frac{1}{2} k \theta^2.$$  

To determine $\omega$ from this equation, we must express the velocities $v_G$, $v_C$, and $\omega_{BC}$ in terms of $\omega$.  

---

(a) Free-body diagram of the system.
Example 19.3  Applying Conservation of Energy to a Linkage (Continued)

Determine Kinematic Relationships  We can determine the velocity of point \( B \) in terms of \( \omega \) and then express the velocity of point \( C \) in terms of the velocity of point \( B \) and the angular velocity \( \omega_{BC} \).

The velocity of \( B \) is

\[
v_B = v_A + \omega_{AB} \times r_{B/A}
\]

\[
= 0 + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega \\ -l \sin \theta & l \cos \theta & 0 \end{bmatrix}
\]

\[
= -l \omega \cos \theta \mathbf{i} - l \omega \sin \theta \mathbf{j}.
\]

The velocity of \( C \), expressed in terms of the velocity of \( B \), is

\[
v_C = v_B + \omega_{BC} \times r_{C/B}
\]

\[
= -l \omega \cos \theta \mathbf{i} - l \omega \sin \theta \mathbf{j} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{BC} \\ l \sin \theta & l \cos \theta & 0 \end{bmatrix}.
\]

Equating \( i \) and \( j \) components, we obtain

\[
\omega_{BC} = -\omega, \quad v_C = -2l \omega \sin \theta.
\]
Example 19.3  Applying Conservation of Energy to a Linkage (Continued)

(The minus signs indicate that the directions of the velocities are opposite to the directions we assumed in Fig. b.) Now that we know the angular velocity of bar BC in terms of \( \omega \), we can determine the velocity of its center of mass in terms of \( \omega \) by expressing it in terms of \( \mathbf{v}_B \):

\[
\mathbf{v}_G = \mathbf{v}_B + \omega_{BC} \times \mathbf{r}_{G/B}
\]

\[
= -l \omega \cos \theta \mathbf{i} - l \omega \sin \theta \mathbf{j} + \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
0 & 0 & -\omega \\
\frac{1}{2} l \sin \theta & \frac{1}{2} l \cos \theta & 0
\end{vmatrix}
\]

\[
= -\frac{1}{2} l \omega \cos \theta \mathbf{i} - \frac{3}{2} l \omega \sin \theta \mathbf{j}.
\]

Substituting these expressions for \( \omega_{BC}, v_C, \) and \( v_G \) into our equation of conservation of energy and solving for \( \omega \), we obtain

\[
\omega = \left[ \frac{2gl(m + m_C)(1 - \cos \theta) - \frac{1}{2} k \theta^2}{\frac{1}{2} ml^2 + (m + 2m_C)l^2 \sin^2 \theta} \right]^{1/2}.
\]

Critical Thinking

Newton’s second law and the equation of angular motion for a rigid body can be applied to this example instead of conservation of energy. How can you decide what approach to use? The energy methods we have described are generally useful only when you can easily determine the work done by forces and couples acting on a system or their associated potential energies. When that is the case, an energy approach is often preferable. To apply Newton’s second law and the equation of angular motion to this example, it would be necessary to draw individual free-body diagrams of the two bars and the collar \( C \), thereby introducing into the formulation the forces exerted on the bars and collar at the pins connecting them. In contrast, we were able to apply conservation of energy to the system as a whole, greatly simplifying the solution.
**RESULTS Section 19.1 (Continued)**

**Conservation of Energy**

If all of the forces that do work on a system of rigid bodies are conservative, the sum of the total kinetic energy and the total potential energy is the same at any two positions.

\[ T_1 + V_1 = T_2 + V_2. \quad (19.18) \]

When both conservative and nonconservative forces do work on a system of rigid bodies, the principle of work and energy can be expressed in terms of the potential energy \( V \) of the conservative forces and the work \( U_{12} \) done by nonconservative forces.

\[ T_1 + V_1 + U_{12} = T_2 + V_2. \quad (19.19) \]

Applying energy methods to a rigid body or system of rigid bodies typically involves three steps.

1. **Identify the forces and couples that do work.** Use free-body diagrams to determine which external forces and couples do work on the system.

2. **Apply the principle of work and energy or conservation of energy.** Either equate the total work done during a change in position to the change in the kinetic energy, or equate the sum of the kinetic and potential energies at two positions.

3. **Determine kinematic relationships.** To complete the solution, it will often be necessary to obtain relations between velocities of points of rigid bodies and their angular velocities.
RESULTS Section 19.1 (Continued)

Power

The power transmitted to a rigid body by a force. The term $v_p$ is the velocity of the point of application of the force.

$$P = F \cdot v_p.$$  \hspace{1cm} (19.20)

The power transmitted to a rigid body by a couple is the product of the couple and the angular velocity of the rigid body.

$$P = M\omega.$$  \hspace{1cm} (19.21)

The average power transferred to a rigid body during an interval of time is equal to the change in kinetic energy of the body, or the total work done during that time, divided by the interval of time.

$$P_{av} = \frac{T_2 - T_1}{t_2 - t_1} = \frac{U_{12}}{t_2 - t_1}. \hspace{1cm} (19.22)$$
The principle of linear impulse and momentum states that the linear impulse applied to a rigid body is equal to the change in its linear momentum.

\[ \int_{t_1}^{t_2} \sum F \, dt = m v_2 - m v_1. \]  

By introducing the average of the total force with respect to time from \( t_1 \) to \( t_2 \),

\[ \sum F_{av} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \sum F \, dt, \]

the principle of impulse and momentum can be expressed in terms of the average force.

\[ (t_2 - t_1) \sum F_{av} = m v_2 - m v_1. \]

If the only forces acting on two rigid bodies \( A \) and \( B \) are the forces they exert on each other, or if other forces are negligible, their total linear momentum is conserved.

\[ m_A v_A + m_B v_B = \text{constant}. \]
Angular Momentum of a Rigid Body in Planar Motion

Angular momentum about the center of mass.

\[ H = I_\omega. \]  \hspace{1cm} (19.27)

One form of the principle of angular impulse and momentum states that the angular impulse about the center of mass during an interval of time from \( t_1 \) to \( t_2 \) is equal to the change in the angular momentum about the center of mass.

\[ \int_{t_1}^{t_2} \sum M \, dt = H_2 - H_1. \]  \hspace{1cm} (19.28)

Equation (19.28) can be expressed in terms of the average moment about the center of mass.

\[ (t_2 - t_1) \Sigma M_{av} = H_2 - H_1. \]  \hspace{1cm} (19.32)
Angular momentum about a fixed point $O$.  

\[ H_O = (\mathbf{r} \times m\mathbf{v}) \cdot \mathbf{k} + I\omega. \]  \hspace{1cm} (19.30)

The term $(\mathbf{r} \times m\mathbf{v}) \cdot \mathbf{k}$ can be evaluated by calculating the “moment” of the angular momentum about point $O$.  

Angular momentum $H_O = D(m|\mathbf{v}|) + I\omega$.  

Angular momentum $H_O = -D(m|\mathbf{v}|) + I\omega$.  

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A second form of the principle of angular impulse and momentum states that the angular impulse about a fixed point \( O \) during an interval of time from \( t_1 \) to \( t_2 \) is equal to the change in the angular momentum about \( O \).

Equation (19.31) can be expressed in terms of the average moment about point \( O \).

If the only forces acting on two rigid bodies \( A \) and \( B \) are the forces they exert on each other, or if the moment due to other forces about a fixed point \( O \) is negligible, the total angular momentum of \( A \) and \( B \) about \( O \) is conserved.

\[
\int_{t_1}^{t_2} \sum M_O \, dt = H_{O2} - H_{O1},
\]

(19.31)

\[
(t_2 - t_1)(\sum M_O)_{av} = H_{O2} - H_{O1},
\]

(19.33)

\[
H_{OA} + H_{OB} = \text{constant}.
\]

(19.34)
RESULTS Section 19.3

Suppose that two rigid bodies $A$ and $B$ in planar motion collide.

**Linear Momentum**

If other forces are negligible in comparison to the impact forces $A$ and $B$ exert on each other, their total linear momentum is the same before and after the impact.
RESULTS Section 19.3 (Continued)

Angular Momentum

If other forces are negligible in comparison to the impact forces $A$ and $B$ exert on each other, their total angular momentum about any fixed point $O$ is the same before and after the impact. If, in addition, $A$ and $B$ exert forces on each other only at their point of impact $P$, the angular momentum about $P$ of each rigid body is the same before and after the impact.

If one of the two rigid bodies has a pin support at a point $O$, their total angular momentum about $O$ is the same before and after the impact.
Let $P$ be the point of impact of rigid bodies $A$ and $B$, and let their velocities at $P$ be $v_{AP}$ and $v_{BP}$ just before the impact and $v'_{AP}$ and $v'_{BP}$ just afterward. The components of velocities perpendicular to the plane of the impact are related by the coefficient of restitution.

$e = \frac{(v'_{BP})_x - (v'_{AP})_x}{(v_{AP})_x - (v_{BP})_x}$.  \hfill (19.35)
Active Example 19.4  Principle of Angular Impulse and Momentum (Related Problem 19.55)

Prior to their contact, disk $A$ has a counterclockwise angular velocity $\omega_0$ and disk $B$ is stationary. At $t = 0$, disk $A$ is moved into contact with disk $B$. As a result of friction, the angular velocity of $A$ decreases and the clockwise angular velocity of $B$ increases until there is no slip between the disks. What are their final angular velocities $\omega_A$ and $\omega_B$? The moments of inertia of the disks are $I_A$ and $I_B$.

Given:

$\omega_0, I_A, I_B$
Initial contact at $t = 0$

Need to find:

$\omega_A, \omega_B$.

Assumptions:

1) Initial slip, then no slip

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Active Example 19.4  Principle of Angular Impulse and Momentum (Continued)

Strategy
The disks rotate about fixed axes through their centers of mass while they are in contact, so we can apply the principle of angular impulse and momentum in the form given by Eq. (19.28) to each disk. When there is no longer any slip between the disks, their velocities are equal at their point of contact. With this kinematic relationship and the equations obtained from the principle of angular impulse and momentum, we can determine the final angular velocities.

Solution

Draw the free-body diagrams of the contacting disks. \( N \) and \( f \) are normal and friction forces they exert on each other.
Active Example 19.4  Principle of Angular Impulse and Momentum (Continued)

Let \( t_f \) be the time at which slip ceases. Apply the principle of angular impulse and momentum to disk \( A \) from \( t = 0 \) to \( t = t_f \), treating counterclockwise moments and angular velocities as positive.

\[
\int_{t_1}^{t_2} \Sigma M \, dt = H_2 - H_1:
\]
\[
\int_{0}^{t} -R_A f \, dt = I_A \omega_A - I_A \omega_0.
\]  
(1)

Apply the principle of angular impulse and momentum to disk \( B \) from \( t = 0 \) to \( t = t_f \), treating counterclockwise moments and angular velocities as positive.

\[
\int_{t_1}^{t_2} \Sigma M \, dt = H_2 - H_1:
\]
\[
\int_{0}^{t} -R_B f \, dt = -I_B \omega_B - 0.
\]  
(2)

Divide Eq. (1) by Eq. (2).

\[
\frac{R_A}{R_B} = \frac{I_A \omega_A - I_A \omega_0}{-I_B \omega_B}.
\]  
(3)

When slip ceases, the velocities of the disks are equal at their point of contact.

\[
R_A \omega_A = R_B \omega_B.
\]  
(4)
Active Example 19.4  
Principle of Angular Impulse and Momentum (Continued)

\[
\omega_A = \left[ \frac{1}{1 + \frac{R_B^2 I_B}{R_A^2 I_A}} \right] \omega_0
\]

\[
\omega_B = \left[ \frac{R_A/R_B}{1 + \frac{R_B^2 I_B}{R_A^2 I_A}} \right] \omega_0
\]

Solve Eqs. (3) and (4) for \(\omega_A\) and \(\omega_B\).

**Practice Problem**  Disk A has an initial counterclockwise angular velocity \(\omega_0\). Before disk A comes into contact with disk B, suppose that you want disk B to have an initial angular velocity such that, when slip between the two disks ceases, their angular velocity is zero. What is the necessary initial angular velocity of disk B?

**Answer:** \(\frac{R_B I_A}{R_A I_B} \omega_0\) counterclockwise.
Example 19.6  Conservation of Angular Momentum (Related Problem 19.58)

In a well-known demonstration of conservation of angular momentum, a person stands on a rotating platform holding a mass \( m \) in each hand. The combined moment of inertia of the person and platform about their axis of rotation is \( I_p = 0.4 \text{ kg} \cdot \text{m}^2 \) and each mass \( m = 4 \text{ kg} \). Neglect the moments of inertia of the masses she holds about their centers of mass—that is, treat them as particles. If the person’s angular velocity with her arms extended to \( r_1 = 0.6 \text{ m} \) is \( \omega_1 = 1 \text{ revolution per second} \), what is her angular velocity \( \omega_2 \) when she pulls the masses inward to \( r_2 = 0.2 \text{ m} \)? (You have observed skaters using this phenomenon to control their angular velocity in a spin by altering the positions of their arms.)
Example 19.6  Conservation of Angular Momentum (Continued)

Strategy
If we neglect friction in the rotating platform, the total angular momentum of the person, platform, and masses about the axis of rotation is conserved. We can use this condition to determine $\omega_2$.

Solution
The total angular momentum is the sum of the angular momentum of the person and platform and the angular momentum of the two masses. When her arms are extended, the velocity of each mass $m$ is $r_1 \omega_1$, so the “moment” of the angular momentum of each mass about the axis of rotation is $r_1 (mr_1 \omega_1)$. The total angular momentum is

$$H_{O1} = I_P \omega_1 + 2r_1 (mr_1 \omega_1).$$

When she pulls the masses inward, the total angular momentum is

$$H_{O2} = I_P \omega_2 + 2r_2 (mr_2 \omega_2).$$
Example 19.6  Conservation of Angular Momentum (Continued)

The total angular momentum is conserved.

\[ H_{O1} = H_{O2}: \]
\[
(I_P + 2mr_1^2)\omega_1 = (I_P + 2mr_2^2)\omega_2,
\]

\[
[0.4 \text{ kg-m}^2 + 2(4 \text{ kg})(0.6 \text{ m})^2]\omega_1 = [0.4 \text{ kg-m}^2 + 2(4 \text{ kg})(0.2 \text{ m})^2]\omega_2.
\]

This yields \( \omega_2 = 4.56\omega_1 = 4.56 \) revolutions per second.

Critical Thinking
Calculate the total kinetic energy of the person, platform, and masses when her arms are extended and when she has pulled the masses inward. You will find that the total kinetic energy is greater in the second case. It appears that conservation of energy is violated. However, she does work on the weights in pulling them inward. Her physiological energy supplies the additional kinetic energy.

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Active Example 19.7  Impact of a Sphere and a Suspended Bar (Related Problem 19.70)

The ball of mass $m_A$ is translating with horizontal velocity $v_A$ when it strikes the stationary slender bar of mass $m_B$ and length $l$. The coefficient of restitution of the impact is $e$.

(a) What is the angular velocity of the bar after the impact?
(b) If the duration of the impact is $\Delta t$, what average horizontal force is exerted on the bar by the pin support $C$ as a result of the impact?

**Strategy**

(a) The total angular momentum about $C$ of the ball and bar is the same before and after the impact. The coefficient of restitution relates the velocities of the ball and the bar *at the point of impact* before and after the impact. With these two equations and kinematic relationships, we can determine the velocity of the ball and the angular velocity of the bar after the impact.

(b) We can determine the average force exerted on the bar by the support at $C$ by applying the principle of angular impulse and momentum to the bar.
Active Example 19.7 Impact of a Sphere and a Suspended Bar (Continued)

(a) Apply conservation of angular momentum about C. After the impact, $v_A'$ is the velocity of the ball, $v_B'$ is the velocity of the center of mass of the bar, and $\omega_B'$ is the angular velocity of the bar. $I_B$ is the moment of inertia of the bar about its center of mass.

\[
\begin{align*}
H_{CA} + H_{CB} &= H'_{CA} + H'_{CB} \\
\frac{1}{2} m_A v_A' + \frac{1}{2} I_B \omega_B' &= \frac{1}{2} m_A v_A' + \frac{1}{2} I_B \omega_B' + \frac{1}{2} I_B \omega_B' \\
1000 kg \cdot m^2/s^2 + 1000 kg \cdot m^2/s^2 &= 1000 kg \cdot m^2/s^2 + 1000 kg \cdot m^2/s^2 + 1000 kg \cdot m^2/s^2
\end{align*}
\]

Apply the coefficient of restitution. After the impact, $v_B'$ is the velocity of the bar at the point of impact.

\[
e = \frac{v_B' - v_A'}{v_A' - 0}.
\]
Active Example 19.7  Impact of a Sphere and a Suspended Bar (Continued)

Determine kinematic relationships. Because it rotates about the fixed point C, the velocities of the bar at its center of mass and at the point of impact can be expressed in terms of the bar’s angular velocity.

\[
\begin{align*}
v_B' &= \frac{1}{2} I \omega_B', \\
v_{BP}' &= h \omega_B',
\end{align*}
\]

Solve Eqs. (1) through (4) for \(v_A', \ v_B', \ v_{BP}'\), and \(\omega_B'\) and use relation \(I_B = \frac{1}{12} m_B l^2\) to obtain the bar’s angular velocity.

\[
\omega_B' = \frac{(1+e) h m_A v_A'}{h^2 m_A + \frac{1}{3} m_B l^2}.
\]
Active Example 19.7  Impact of a Sphere and a Suspended Bar (Continued)

Apply the principle of angular impulse and momentum in the form given by Eq. (19.33) about point $P$ where the impact occurs.

\[
(t_2 - t_1)(\Sigma M_P)_a = H_B - H_B:
\]

\[
\Delta t \left( hC_c \right) = - \left( h - \frac{1}{2} l \right) m_B v_B' + I_B \omega_B' = 0. \quad (6)
\]

Solve Eq. (6) for $C_c$ and use Eqs. (3) and (5) and the relation $I_B = \frac{1}{12} m_B l^2$.

\[
C_c = \frac{(1 + e) \left( \frac{1}{2} h - \frac{1}{3} l \right) I_m m_B v_B}{\left( h^2 m_A + \frac{1}{3} m_B l^2 \right) \Delta t}.
\]

**Practice Problem** Suppose that the pin support at $C$ is removed, and the ball strikes the stationary vertical bar with horizontal velocity $v_A$. Assume that $m_A = m_B$ and $h = \frac{3}{4} l$. What is the angular velocity of the bar after the impact?

**Answer:** $\omega_B' = \frac{12(1 + e) v_A}{l}$. 

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